Properties of limits

Theorem (Limits rules). If L, M, c, and k are real numbers and

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

Sum Rule	$\lim_{x \to c}(f(x) + g(x)) = L + M$
Difference Rule	$\lim_{x \to c}(f(x) - g(x)) = L - M$
Product Rule	$\lim_{x \to c}(f(x)g(x)) = LM$
Constant Rule	$\lim_{x \to c} kf(x) = kL$
Quotient Rule	$\lim_{x \to c}(f(x)/g(x)) = \frac{L}{M},\ M \neq 0$
Power Rule	$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$, provided that $L^{r/s}$ is a real number

Example 1. Find the following limits

1. $\lim_{x\to c}(x^3+4x^2-3)$

2. $\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}$

3. $\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}$

Example 2. Evaluate

1. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$

2. $\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}$

Theorem (Sandwich Theorem). Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then

$$\lim_{x\to c} f(x) = L$$

Example 3. 1) Given that $1-\frac{x^2}{4} \leq u(x) \leq 1+\frac{x^2}{2}$. Show that $\lim_{x\to 0} u(x) = 1$.

2) If $\lim_{x\to c} |f(x)| = 0$, show $\lim_{x\to c} f(x) = 0$

One sided limit

Definition. Let $f(x)$ be defined on an interval (a, b), where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has **a right-hand limit** L at a and we write

$$\lim_{x\to a^+} f(x) = L$$

Let $f(x)$ be defined on an interval (c, d), where $c < d$. If $f(x)$ approaches arbitrarily close to M as x approaches d from within that interval, then we say that f has **a left-hand limit** M at d and we write

$$\lim_{x\to d^-} f(x) = M$$

Example 4. Graph the function $f(x) = \frac{x}{|x|}$ and find the right and left hand limits at 0.

Theorem. A function $f(X)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one sided limits are equal.

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L$$

Example 5. Using $\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$ for $0 < \theta < \frac{\pi}{2}$ and using the sandwich theorem we get

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Example 6. Find the following limits

1. $\lim_{h \to 0} \frac{\cos h - 1}{h}$

2. $\lim_{x \to 0} \frac{\sin 2x}{5x}$