Trigonometric functions

Radian Measure: The radian measure of the angle ACB at the center of the unit circle equals the length of the arc that ACB cuts from the unit circle. When an angle of measure θ is placed in standard position at the center of a circle of radius r, the six basic trigonometric functions of θ are defined as follows:

$$\begin{aligned}
\textbf{sine}: &\quad \sin\theta = \frac{y}{r}\\
\textbf{cosecant}: &\quad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}\\
\textbf{cosine}: &\quad \cos\theta = \frac{x}{r}\\
\textbf{secant}: &\quad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}\\
\textbf{tangent}: &\quad \tan\theta = \frac{y}{x}\\
\textbf{cotangent}: &\quad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}
\end{aligned}$$

Graph of trigonometric functions

Periodicity

Definition. A function $y = f(x)$ is **periodic** if there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Example 1.

Period π	$\tan(x+\pi) = \tan x$
	$\cot(x+\pi) = \cot x$
Period 2π	$\sin(x+2\pi) = \sin x$
	$\cos(x+2\pi) = \cos x$
	$\sec(x+2\pi) = \sec x$
	$\csc(x+2\pi) = \csc x$

Even and odd trigonometric functions : Show that $\sin x$, $\tan x$, $\csc x$ and $\cot x$ are odd functions and that $\cos x$ and $\sec x$ are even functions.

Transformation of trigonometric functions

$$y = af(b(x+c)) + d$$

a	Vertical stretch or shrink; reflection about x-axis
b	Horizontal stretch or shrink;reflection about y-axis
c	Vertical shift
d	Horizontal shift

Example 2. Graph the function $f(x) = A\sin\left[\frac{2\pi}{B}(x-C)\right] + D$

Some identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$c^2 = a^2 + b^2 - 2ab\cos\theta$

Inverse Trigonometric functions

Example 3. Show that the function $y = \sin x$, $-\pi/2 \leq x \leq \pi/2$, is one-to-one, and graph its inverse.

Definition (Inverse trigonometric functions).

Function	Domain	Range
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \sec^{-1} x$	$\|x\| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
$y = \csc^{-1} x$	$\|x\| \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

Graph of inverse trigonometric functions

The inverse trigonometric functions satisfy the following

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

Common values of inverse trigonometric functions

Example 4. (a) $\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$

(b) $\sin^{-1}(-\frac{1}{\sqrt{2}}) = \frac{-\pi}{4}$

(c) $\cos^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$

(d) $\cos^{-1}(-\frac{1}{2}) = \frac{2}{3}\pi$

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Some Identities

1. $y = \sin^{-1} x$ is an odd function, so

$\sin^{-1}(-x) = -\sin^{-1} x$

2. $\cos^{-1} x + \cos^{-1}(-x) = \pi$

3. $\sin^{-1} x + \cos^{-1} x = \pi/2$